Math 259A Lecture 6 Notes

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1 GNS Construction and Topologies on $\mathcal{B}(H)$

1.1 Every C^*-algebra is an operator algebra

Recall the GNS construction: Let M be a C^* algebra, and let φ be a positive functional. We get a Hilbert space H_φ with $\langle x, y \rangle_\varphi = \varphi(y^* x)$ and a representation $\pi_\varphi(x)(\hat{y}) = \widehat{xy}$.

Remark 1.1. Our representation uses left multiplication, but there is no preference. If we take $\langle x, y \rangle = \varphi(xy^*)$, then we could use right multiplication for our representation π.

This construction gives us $\xi_\varphi = \hat{1}_M$. This gives $H_\varphi = \overline{s\pi_\varphi(M)\xi_\varphi}$. We call ξ_φ a **cyclic vector** for the representation π_φ.

Lemma 1.1. *If H_i is a Hilbert space for all i and $T_i \in B(H_i)$ with $\sup \|T_i\| < \infty$, then $\bigoplus_i T_i \in \mathcal{B}(H_i)$, where $(\bigoplus_i T_i)(\bigoplus_i \xi_i) = \bigoplus_i T(\xi_i)$.*

Theorem 1.1 (GNS)**.** *If M is a C^*-algebra, there exists an isometric, unital, *-algebra morphism $\pi : M \to \mathcal{B}(H)$, where H is a Hilbert space.*

Proof. If $\pi_i : M \to B(H_i)$ are representations for all i, we can define $\pi = \bigoplus_i \pi_i : M \to \mathcal{B}(\bigoplus_i H_i)$ by $\pi(x) = \bigoplus_i \pi_i(x)$. The inner product on $\bigoplus_i H_i$ is $\langle (\xi_i)_i, (\eta_i)_i \rangle = \sum_i \langle \xi_i, \eta_i \rangle_{H_i}$. Injective implies isometric, so it suffices to find a 1 to 1 representation. So it suffices to show that for any $x \neq 0$ in M, there exists $\pi_x : M \to B(H_x)$ such that $\pi_x(x) \neq 0$. By the GNS construction, it suffices to get a positive functional φ on M such that $\|\pi_\varphi(x)\hat{1}\| = \varphi(x^*x) \neq 0$.

The subspace M_+ is closed and convex in M_h and does not contain $-x^*x$. By Hahn-Banach, there exists a continuous $\varphi : M_h \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ such that $\varphi(M_+) < [\alpha, \infty)$ and $\varphi(-x^*x) < \alpha$. Since $0 \in M_+$, $0 \in [\alpha, \infty)$, making $\alpha \leq 0$. If $\alpha > 0$, then $\lambda y \in M_+ \implies \varphi(\lambda y) < 0$. This is a contradiction, so $\alpha = 0$. So φ is positive and $\varphi(x^*x) \neq 0$. $\square$

Remark 1.2. To get the isometry property, we could have produced a φ such that $\varphi(x^*x) = \|x^*x\|$.

1.2 Topologies on $\mathcal{B}(H)$

If H is a Hilbert space, we have multiple choices for norms on $\mathcal{B}(H)$.

Definition 1.1. The **operator norm topology** is the norm topology given by

$$\|T\| = \sup_{\xi \in (H)_1} \|T\xi\| = \sup_{\xi,\eta \in (H)_1} |\langle T\xi, \eta\rangle|.$$

Definition 1.2. The **weak operator topology** is the topology generated by the seminorms $T \mapsto |\langle T\xi, \eta\rangle|$ for all $\xi, \eta \in H$.

Definition 1.3. The **strong operator topology** is the topology generated by the seminorms $T \mapsto \|T\xi\|$ for all $\xi \in H$.

The WOT is weaker than the SOT, which is weaker than the NT.

Definition 1.4. A **von Neumann algebra** is a *-algebra $M \subseteq B(H)$ with $1_M = \mathrm{id}_M \in M$ which is closed in the weak operator topology.

So every von Neumann algebra is a C^*-algebra.

Definition 1.5. Let X be a Banach space, and let $Y \subseteq X^*$ be a vector subspace. The $\sigma(X, Y)$ **topology** on X is given by the seminorms $x \mapsto |\varphi(x)|$ for $\varphi \in Y$.

Proposition 1.1. *Let X be a Banach space, and let $Y \subseteq X^*$ be a vector subspace.*

1. *A linear functional $\varphi : X \to \mathbb{C}$ is $\sigma(X, Y)$-continuous if and only if $\varphi \in Y$.*
2. *A linear functional $\varphi : X \to \mathbb{C}$ is $\sigma(X, Y)$-continuous on $(X)_1$ iff $\varphi \in \overline{Y} \subseteq X^*$ (closure with respect to the norm topology).*
3. *The topologies $\sigma(X, Y)$ and $\sigma(X, \overline{Y})$ coincide on $(X)_1$.*
4. *If $\overline{Y} = Y$, a linear functional φ is $\sigma(X, Y)$-continuous if and only if it is $\sigma(X, Y)$-continuous on $(X)_1$.*

Denote by $B_\sim = \mathrm{span}\{\omega_{\cdot\xi,\eta} = \langle \xi, \eta\rangle : \xi, \eta \in H\} \subseteq \mathcal{B}^*$, and denote $\mathcal{B}_* = \overline{\mathcal{B}_\sim} \subseteq \mathcal{B}^*$.

Remark 1.3. The weak operator topology is the $\sigma(\mathcal{B}, \mathcal{B}_\sim)$ topology on $\mathcal{B}(H)$.

Remark 1.4. Let $FR \subseteq \mathcal{B}(H)$ be the space of finite rank operators. Then $FR \to \mathcal{B}_\sim$ given by $T \mapsto \omega_T$, where $\omega_T(x) = \mathrm{tr}_{\mathcal{B}(H)}(xT)$ is an isomorphism.

Definition 1.6. The **ultraweak toplogy** on $\mathcal{B}(H)$ is the $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology.

Corollary 1.1. *Let $\mathcal{B} = \mathcal{B}(H)$ for a Hilbert space H.*

1. *$\mathcal{B}_\sim$ is the space of weak operator continuous functionals on $\mathcal{B}(H)$.*

2. $\mathcal{B}_*$ *is the space of ultraweak continuous functionals functionals on* $\mathcal{B}$

3. $\varphi : B \to \mathbb{C}$ *is ultraweak continuous if and only if it is weak operator continuous on* $(\mathcal{B})_1$.

4. *The weak and ultraweak topologies coincide on* $(\mathcal{B})_1$.

Theorem 1.2. $\varphi : \mathcal{B} \to \mathbb{C}$ *is weak operator continuous if and only if it is it is strong operator continuous.*